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Dade's Projective Conjecture for p-Solvable Groups

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INTRODUCTION

At MSRI, in late 1990, E. C. Dade stated his ordinary and projective conjectures and announced that strengthened versions of these conjectures could be made which were amenable to induction and could be reduced to questions about simple groups. See also [3–5]. The conjectures concern numbers of complex irreducible characters in certain p-blocks for a prime number p.

At present, it is our understanding that a statement of the "final" form of the "inductive" conjecture which admits the reduction mentioned above has not yet been published (Dade has informed us that the version which appeared in [5] needs further refinement). It also seems likely that the reductions announced by Dade will not appear for several more years.

In view of this, we feel that it would be a service to certain sections of the representation-theoretic community to present a self-contained proof of the conjecture of the title for p-solvable groups, as there is no published proof of this case currently available (or even a preprint version, as far as we are aware).

The reader should take note of the fact that p-solvable groups are more amenable to the Clifford-theoretic reductions of block theory than are general finite groups (although they already throw up problems of formidable difficulty—for example, we are fortunate here to be able to assume the results of Külshammer and Puig [10] as one of our starting points) and should not be misled into believing that the necessary Clifford-theoretic reductions in the general case will be as straightforward.

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1. STATEMENT OF THE CONJECTURE

Let G be a finite group, p a rational prime number, and $(\mathbb{K}, R, F)$ a "sufficiently large" p-modular system. This includes the requirement that R is a complete discrete valuation ring of characteristic 0, with field of fractions $\mathbb{K}$, such that $F = R/J(R)$ is algebraically closed of characteristic p. We also require that $\mathbb{K}$ contain a primitive $|G|_p$th root of unity. As usual, for a non-zero integer n and a prime number q, the highest power of q dividing n is denoted by n_q. The hypotheses on F and R imply that $\mathbb{K}$ contains p'-roots of unity of all orders. Hence we may, and do, suppose that $\mathbb{K}$ contains all p'-roots of unity in $\mathbb{C}$, the complex root of unity $e^{2\pi i/|G|_p}$, and the rational field $\mathbb{Q}$.

Let X be a finite p'-central extension of a section of G. Then the Krull–Schmidt theorem holds for finitely generated RX modules, as R is complete. We may identify complex characters of X with $\mathbb{K}$-valued characters of X. Whenever we speak of characters from now on, we will be referring to complex characters. As is now customary, we say that an irreducible character, χ, of X with $p^d\chi(1)_p = |X|_p$ has defect d.

We will sometimes refer to blocks of the group algebra RX simply as "p-blocks of X." Let b be a block of RX. We denote the set of irreducible characters in b by $\mathrm{Irr}(b)$. We denote the multiplicative identity of b by 1_b.

Now let Y be a central p-subgroup of X. We denote by $\mathscr{N}(X, Y)$ the set of all normal chains of p-subgroups of X of the form

$$\sigma = (Q_0(=Y) < Q_1 < \cdots < Q_n),$$

where $Q_i \lhd Q_n$ for $i = 1, \dots, n$ (we include the singleton chain (Y)). For such a chain σ, we let $|\sigma| = n$, the number of p-subgroups appearing in σ which strictly contain Y (this is consistent with Knörr and Robinson [9] in the case that Y is trivial). We let V^σ denote the maximal p-subgroup appearing in σ and let X_σ denote the stabilizer of σ under the natural conjugation action of X on $\mathscr{N}(X, Y)$. We define the sum of blocks b_σ of RX_σ as in [9] (as we are dealing with normal chains, this is just the sum of the usual Brauer correspondents of b for X_σ). As is now customary, the notation $L < M$ for finite groups L and M indicates that L is a proper subgroup of M.

For b' a sum of blocks of RX and d a non-negative integer, we let $k_d(b')$ denote the number of irreducible characters of defect d in b'. More generally, for γ a linear character of Y, we denote the number of irreducible characters of defect d in b' which lie over γ by $k_d(b', \gamma)$.

For a p-subgroup, Q, of X, we let Br_Q denote the composition of the natural epimorphism from RX to FX followed by the vector space projection from FX onto $FC_X(Q)$ with kernel $F[X \setminus C_X(Q)]$. Then Br_Q

restricts to a homomorphism of R-algebras from RX^Q (the subalgebra of RX consisting of the Q-fixed points under conjugation action) to $FC_X(Q)$.

We suppose from now on that $O_p(G)$, the unique largest normal p-subgroup of G, is contained in the center, $Z(G)$, of G. We set $Z = O_p(G)$ and we fix a block, B, of RG whose defect groups are not central in G.

Dade's projective conjecture, in the version we prefer to treat, asserts that

$$\sum_{\sigma \in \mathcal{N}(G, Z)/G} (-1)^{|\sigma|} k_d(B_\sigma, \lambda) = 0$$

for each linear character, λ, of Z and every positive integer d. The notation $/G$ indicates that we are taking a full set of representatives for the G-orbits under the given action.

Remarks. The reader may wonder why we only seem to consider irreducible characters lying over linear characters of central p-subgroups. We are dealing with a "blockwise" conjecture so, for a linear character, μ, of a central p'-subgroup, W, of G, either all irreducible characters in B (and, in fact, in every B_σ) lie over μ or none do. The reader may also wonder why we do not use radical chains. As explained in [9], radical chains may be interchanged with chains associated to various p-subgroup complexes (strictly, we are identifying chains which begin with Z with chains from the p-subgroup complex of G/Z). It will become apparent that normal chains are more suitable for the proof of our main theorem.

The following observation (noted by Hassan and Horvath in special cases in [8]) is useful. Suppose that the formula

$$\sum_{\overline{\sigma} \in \mathcal{N}(\overline{G}, \overline{Z})/\overline{G}} (-1)^{|\sigma|} k_{d'}\left(\overline{B}_{\overline{\sigma}}\right) = 0$$

(which we henceforth call the "weak" version of Dade's projective conjecture for $\overline{G}$) holds whenever d' is a positive integer; $\overline{G}$ is a homomorphic image of G of the form G/Z_1 for a subgroup, Z_1, of Z; and $\overline{B}$ is the unique block of $R\overline{G}$ which is the image of B under the natural epimorphism from RG to $R\overline{G}$. Then

$$\sum_{\sigma \in \mathcal{N}(G, Z)/G} (-1)^{|\sigma|} k_d(B_\sigma, \lambda) = 0$$

for each linear character, λ, of Z, and each positive defect d.

We outline a justification. Fix a choice of such d and λ. By induction, if λ is not faithful we may suppose that the appropriate alternating sum (with $d' = d - \log_p(|\ker \lambda|)$ in place of d) is 0 for $\overline{G} = G/\ker \lambda$. Hence we may, and do, suppose that λ is faithful and that Z is cyclic. Next, considering

$G/\Omega_1(Z)$, we may delete from the alternating sum of the "weak" version (for G) the contribution from characters which lie over non-faithful linear characters of Z and still obtain 0. As usual, $\Omega_1(Z)$ denotes the subgroup of Z generated by the elements of order p in Z. All faithful linear characters of Z are p-conjugate, so that the number of irreducible characters of fixed defect of any B_σ which lie over any one of them is independent of the particular faithful linear character chosen.

2. PROOF OF THE THEOREM

THEOREM 1. *Suppose that the formula appearing in Dade's projective conjecture fails to hold (for some defect, d, and some linear character, λ, of the central subgroup $Z = O_p(G)$) for the block B of RG, and that first $[G : Z(G)]$ and then $|G|$ have been minimized subject to such a failure occurring. Then $O_{p'}(G)$, the unique largest normal p'-subgroup of G, is central in G.*

Theorem 1 (or, more precisely, its proof) yields a proof of Dade's projective conjecture for p-solvable groups. A p-solvable finite group H such that $O_p(H)O_{p'}(H) \leq Z(H)$ is Abelian. Dade's projective conjecture is vacuously true for Abelian groups, so the proof of Theorem 1 (with the arguments restricted to p-solvable groups) shows that no p-solvable counterexample to Dade's projective conjecture exists.

We will have occasion to deal with Morita equivalences between p-blocks of various finite groups. These Morita equivalences (which are for blocks defined over R) induce defect-preserving bijections between the sets of irreducible characters in the equivalent blocks (one explanation of this is that such Morita equivalent blocks have the same decomposition matrix D (under a suitable labeling), so they have the same Cartan matrix C. The matrix $DC^{-1}D^T$ (which is a scalar multiple of the famous Brauer−Feit matrix) determines the number of irreducible characters of each defect in the equivalent blocks, since its rows may be used to index the irreducible characters in either block, and the defect of the irreducible character (for either block) corresponding to its ith row is d_i, the smallest non-negative integer such that p^{d_i} times the ith row has integer entries). In the case of Morita equivalences obtained via Fong correspondence, this defect-preserving bijection can also be seen directly from the usual character-theoretic constructions.

Before we proceed to the proof of Theorem 1, we mention some consequences. C. Eaton, in [6], has recently proved that Dade's projective conjecture implies Conjecture 4.1 of [12] (the reverse implication being immediate). Hence Conjecture 4.1 of [12] is valid for p-solvable groups, so that the consequences of that conjecture derived in [12] are valid for such groups. Perhaps the most interesting of these is:

THEOREM 2. *Let H be a finite p-solvable group and b be a p-block of H with defect group D. Suppose that b contains an irreducible character of defect d', lying over the linear character γ of the central p-subgroup, W, of H. Then there is a subgroup, U, of D with $U = O_p(N_H(U))$ and $C_D(U) = Z(U)$, and there is an irreducible character of U, lying over γ, which has defect d'.*

Remark. L. Barker, in [1], proves a result implying most of Theorem 2 (except that it is not clear that the p-subgroup he produces is radical, so that his result does not seem to give the full strength of the p-solvable case of the consequence of Conjecture 4.1 of [12] stated in Theorem 5.1 of [12]).

The proof of the p-solvable case of Dade's projective conjecture yields another proof of the p-solvable case of Alperin's weight conjecture (consider the case that Z is trivial, sum over all defects d, and use the formulation of Alperin's weight conjecture given in [9]). We take the opportunity to point out that the "sketch" proof of Alperin's weight conjecture for p-solvable groups given in Robinson and Staszewski [11] is not invalid (as L. Barker, in [2], asserts) since it invoked the results of Külshammer and Puig [10] (then in preprint form).

Proof of Theorem 1. Let G, B, Z, λ, and d be as in the statement of the theorem. Suppose further that $O_{p'}(Z(G)) < N$ where $N \lhd G$ and that N has order prime to p.

By the minimality of G and the discussion at the end of Section 1, the formula of the "weak" form of the conjecture fails for B with our chosen defect d. Let μ be an irreducible character of N which lies under B. Whenever Q is a p-subgroup of the inertial subgroup, $I_G(\mu)$, of μ in G, we let $f_Q(\mu)$ denote the Glauberman correspondent of μ for $C_N(Q)$.

Suppose first that μ is G-stable. Then, as usual, we may construct a finite p'-central extension, say $\hat{G}$, of G, using the projective representation of G associated to μ. Then $\hat{G}$ may be assumed to have a finite cyclic central p'-subgroup $\hat{W}$ such that $\hat{G}/\hat{W} \cong G$. Also, $\hat{G}$ may be assumed to have a normal subgroup, $\hat{N}$, naturally isomorphic to N, such that $\hat{N} \cap \hat{W}$ is trivial. Identifying μ with an irreducible character of $\hat{N}$, we may extend μ to an irreducible character, $\hat{\mu}$ say, of $\hat{G}$.

We set $\hat{G}/\hat{N} = \tilde{G}$ and, more generally, we denote the image in $\tilde{G}$ of a subgroup, $\hat{Y}$, of $\hat{G}$ by $\tilde{Y}$. Then $\tilde{G}/\tilde{W} \cong G/N$ and we note that $[\tilde{G} : Z(\tilde{G})] \leq [G : ZN] < [G : Z(G)]$.

Fong correspondence now gives a unique block, $\tilde{B}$, of $R\tilde{G}$ corresponding to, and Morita equivalent to, B. More precisely, let $\hat{\gamma}$ be the unique linear character of $\hat{W}$ covered by $\hat{\mu}$ and identify this naturally with a linear character, $\tilde{\gamma}$, of $\tilde{W}$. Then there is a bijection between the set of blocks of

$R\tilde{G}$ covering the complex conjugate of $\tilde{\gamma}$ and the set of blocks of RG covering μ. Furthermore, given one of the former blocks, inflating its irreducible characters to characters of $\hat{G}$, then multiplying them by $\hat{\mu}$, yields (the inflations to $\hat{G}$ of) the irreducible characters in the corresponding block of RG.

Since N is a p'-group, there is a natural bijection between $\mathscr{N}(G,Z)/G$ and $\mathscr{N}(\tilde{G},\tilde{Z})/\tilde{G}$. Let bars denote images in G/N. First, as $\tilde{W}$ is a central p'-subgroup of $\tilde{G}$ with $\tilde{G}/\tilde{W} \cong \overline{G}$, there is an obvious conjugacy-preserving bijection between $\mathscr{N}(\overline{G},\overline{Z})$ and $\mathscr{N}(\tilde{G},\tilde{Z})$. We now describe in more detail the bijection between $\mathscr{N}(G,Z)/G$ and $\mathscr{N}(\overline{G},\overline{Z})/\overline{G}$.

Clearly, G-conjugate chains in $\mathscr{N}(G,Z)$ map to $\overline{G}$-conjugate chains in $\mathscr{N}(\overline{G},\overline{Z})$. On the other hand, let

$$\bar{\sigma} = \left(\overline{Q}_0(=\overline{Z}) < \overline{Q}_1 < \cdots < \overline{Q}_n\right)$$

be a chain in $\mathscr{N}(\overline{G},\overline{Z})$. For each i, let P_i be the full pre-image of $\overline{Q}_i$ in G. Let Q_n be a Sylow p-subgroup of P_n. For $i < n$, set $Q_i = P_i \cap Q_n$. Then $Q_i \in \mathrm{Syl}_p(P_i)$, as $P_i \lhd P_n$ for each i (in particular, $Q_0 = Z$). Also, $Q_i \lhd Q_n$ for each i, so that we have produced a chain $(Q_0 < Q_1 < \cdots < Q_n)$ in $\mathscr{N}(G,Z)$. We denote this chain by σ. It is not unique, but is uniquely determined by $\bar{\sigma}$ and the choice of Sylow p-subgroups of P_n, so it is determined uniquely up to N-conjugacy by $\bar{\sigma}$.

Any chain $\tau \in \mathscr{N}(G,Z)$ which maps onto $\bar{\sigma}$ under the natural homomorphism from G to $\overline{G}$ has the form $\tau = (Z < Q_1^{x_1} < Q_2^{x_2} < \cdots < Q_n^{x_n})$ with each $x_i \in N$. We are really only concerned with the N-conjugacy class of τ, so we may assume that $x_n = 1_G$ and we do so. Then, for each i, we have $[Q_i, x_i] \leq Q_n \cap N = 1$, so that $\tau = \sigma$. Hence $\bar{\sigma}$ has a unique N-conjugacy class of pre-images in $\mathscr{N}(G,Z)$. In particular, $\overline{G}$-conjugate chains in $\mathscr{N}(\overline{G},\overline{Z})$ "lift" to G-conjugate chains in $\mathscr{N}(G,Z)$. The argument above also shows that $\overline{G}_{\bar{\sigma}} = NG_\sigma/N$.

As a matter of interest, we remark that chains which are radical in $\mathscr{N}(G,Z)$ need not have radical images in $\mathscr{N}(\overline{G},\overline{Z})$ in general, as can be seen already in p-nilpotent examples.

A key observation for the proof of the theorem is that $G_\sigma = N_{NG_\sigma}(V^\sigma)$ for all $\sigma \in \mathscr{N}(G,Z)$. For (using Dedekind's modular law and the fact that N is a normal p'-subgroup of G) we have $N_{NG_\sigma}(V^\sigma) = G_\sigma N_N(V^\sigma) = G_\sigma C_N(V^\sigma) = G_\sigma$.

This observation allows us to employ results from Harris and Knörr [7] and Külshammer and Puig [10]. These results both concern p-blocks of a group X which cover an X-stable p-block, b say, of a normal subgroup, T, of X. For our purposes, X may be taken to be a finite p'-central extension of a section of G. The main result of [7] tells us that, for Q a defect group of b, Brauer correspondence gives a bijection between blocks of RX which lie over b and blocks of $RN_X(Q)$ which lie over the unique Brauer correspondent of b for $RN_T(Q)$. The main result of [10] has the consequence that if the block b is also nilpotent, then there is a Morita equivalence between the corresponding blocks of RX and $RN_X(Q)$.

Here (and later in the paper) we make use of the compatibility (first made explicit by Alperin) between Glauberman correspondence and Brauer correspondence in the case that the coprime group of operators is a p-group.

Whenever σ is a chain in $\mathscr{N}(G, Z)$, each block in B_σ has defect groups containing V^σ and lies over the Glauberman correspondent $f_{V^\sigma}(\mu)$. By [7] (with NG_σ in the role of X and NV^σ in the role of T), each block of RNG_σ which has a defect group containing V^σ and which lies over the unique block of RNV^σ covering μ has a single Brauer correspondent for G_σ. Furthermore, each block of RG_σ which lies over $f_{V^\sigma}(\mu)$ occurs as such a Brauer correspondent. Moreover, as all p-blocks of p-nilpotent groups are nilpotent, two such Brauer correspondent blocks are Morita equivalent, by [10].

Hence, using B_σ, we may define the Morita equivalent sum, B'_σ, of Harris–Knörr correspondent blocks for RNG_σ, with all irreducible characters of B'_σ lying over μ. We may then use Fong correspondence to obtain from B'_σ a Morita equivalent sum of blocks, $\tilde{B}_\sigma$ say, of $R\tilde{G}_{\tilde{\sigma}}$ (note the apparent distinction between this sum of blocks and $\tilde{B}_{\tilde{\sigma}}$).

In fact, $\tilde{B}_\sigma = \tilde{B}_{\tilde{\sigma}}$, but we explain this in more detail, as the fact that Brauer correspondence and Fong correspondence commute in this way is not as widely known as it might be (though the referee informs us that it can be deduced with some effort from Theorem 14.3 of [4]). This part of the argument is quite general and does not rely on the fact that G is a minimal counterexample.

For the sake of exposition, we consider the case that μ extends to an irreducible character of G (in general, we would need to work with the group $\hat{G}$ discussed earlier in the proof). We choose a particular extension of μ, which we denote by $\mu^{(G)}$, to an irreducible character of G. Whenever we are dealing with a subgroup, X, of G which contains N, we set $\mu^{(X)} = \mathrm{Res}^G_X(\mu^{(G)})$, an extension of μ to an irreducible character of X.

We let bars denote images in G/N. We choose a defect group, D, for B. We set $M = N_G(D)$ and $H = NM$. Then $\overline{D}$ is a defect group for the Fong correspondent, $\overline{B}$, of B. By Brauer's First Main Theorem (as H contains M) there is just one Brauer correspondent, b' say, of B for RH which has defect group D. Let b be the unique Brauer correspondent of B for M. Then b' is also the Harris–Knörr correspondent of b for $H = NM$, so that b' lies over μ, as b lies over the Glauberman correspondent $f_D(\mu)$.

We know that $\mathrm{Irr}(B) = \{\chi\mu^{(G)} : \chi \in \mathrm{Irr}(\overline{B})\}$ and that $\mathrm{Irr}(b') = \{\theta\mu^{(H)} : \theta \in \mathrm{Irr}(\overline{b})\}$ for the Fong correspondent block, $\overline{b}$, of b' for $R\overline{H}$

(identifying characters in $\mathrm{Irr}(\overline{B})$ with their inflations to G, and likewise for $\mathrm{Irr}(\overline{b})$ and H).

We wish to prove that $\overline{b}$ is the (unique) Brauer correspondent for $R\overline{H}$ of $\overline{B}$ (note that $\overline{H} = \overline{M} = N_{\overline{G}}(\overline{D})$). We need to establish the usual congruence for central character values on p-regular class sums of $\overline{H}$ with defect group $\overline{D}$.

We claim that for each p-regular $\bar{x}$ with $\overline{D} \in \mathrm{Syl}_p(C_{\overline{G}}(\bar{x}))$ we may find a p-regular pre-image, x, of $\bar{x}$ in H with $D \in \mathrm{Syl}_p(C_G(x))$ such that $\mu^{(G)}(x) \notin J(R)$.

We choose a (necessarily p-regular) pre-image, y say, of $\bar{x}$ in G. Note (as N is a p'-group and $\overline{D}$ is a Sylow p-subgroup of $C_{\overline{G}}(\bar{x})$) that ND contains a Sylow p-subgroup of $C_G(yn)$ for each $n \in N$ and that any such Sylow p-subgroup is N-conjugate to a subgroup of D. Note also that D acts by conjugation on the coset yN. Since $\mu^{(G)}$ restricts irreducibly to N, we know that $\sum_{n \in N} |\mu^{(G)}(yn)|^2 = |N| \notin J(R)$. It follows that there is some $n' \in N$ such that $[D : C_D(yn')]\mu^{(G)}(yn') \notin J(R)$. Hence D must centralize yn' and $\mu^{(G)}(yn') \notin J(R)$. Then yn' is p-regular, $D \in \mathrm{Syl}_p(C_G(yn'))$, and $\mu^{(G)}(yn') \notin J(R)$, so that we may take $x = yn'$.

Now we take a p-regular element $\bar{x}$ in $\overline{H}$ with $\overline{D} \in \mathrm{Syl}_p(C_{\overline{G}}(\bar{x}))$ and we choose a p-regular pre-image, x, of $\bar{x}$ in G of the type just described.

Let T_x denote the full pre-image in G of $C_{\overline{G}}(\bar{x})$. Then $D \in \mathrm{Syl}_p(T_x)$, so that, using Sylow's theorem, we see easily that $[T_x : C_G(x)] \equiv [T_x \cap H : C_H(x)] \not\equiv 0 \pmod{p}$, as $M \leq H$.

Let χ be an irreducible character in $\overline{B}$ and let θ be an irreducible character in $\overline{b}$. Short calculations show that

$$[G : C_G(x)] \frac{\chi(x)\mu^{(H)}(x)}{\chi(1)\mu^{(H)}(1)} = \left[\overline{G} : C_{\overline{G}}(\bar{x})\right] \frac{\chi(\bar{x})}{\chi(1)} [T_x : C_G(x)] \frac{\mu^{(H)}(x)}{\mu^{(H)}(1)}$$

and that

$$[H : C_H(x)] \frac{\theta(x)\mu^{(H)}(x)}{\theta(1)\mu^{(H)}(1)} = \left[\overline{H} : C_{\overline{H}}(\bar{x})\right] \frac{\theta(\bar{x})}{\theta(1)} [T_x \cap H : C_H(x)] \frac{\mu^{(H)}(x)}{\mu^{(H)}(1)}.$$

Also, using Brauer correspondence between B and b', we have

$$[G : C_G(x)] \frac{\chi(x)\mu^{(H)}(x)}{\chi(1)\mu^{(H)}(1)} \equiv [H : C_H(x)] \frac{\theta(x)\mu^{(H)}(x)}{\theta(1)\mu^{(H)}(1)} (\mathrm{mod}\, J(R)).$$

Since $\mu^{(H)}(x)/\mu^{(H)}(1)$ is a unit of R and $[T_x : C_G(x)] \equiv [T_x \cap H : C_H(x)] \not\equiv 0 \pmod{p}$, we conclude that

$$\left[\overline{G} : \overline{C}_{\overline{G}}(\bar{x})\right] \frac{\chi(\bar{x})}{\chi(1)} \equiv \left[\overline{H} : \overline{C}_{\overline{H}}(\bar{x})\right] \frac{\theta(\bar{x})}{\theta(1)} (\operatorname{mod} J(R)).$$

Hence $\overline{b}$ is the (unique) Brauer correspondent for $R\overline{H}$ of $\overline{B}$.

This establishes that Fong correspondence and Brauer correspondence commute in the desired fashion in the case that $\sigma = (Z < D)$, where D is a defect group for B.

We digress for a moment to discuss the behavior of Fong correspondence when we have a pair of subgroups T and X of G, with $N \leq T \lhd X$ and a block, β, of RX, lying over μ, covering blocks $\beta_1, \dots, \beta_n$ (and no others) of RT.

We use bars (over blocks of RX or RT covering μ) to denote their Fong correspondents for $\overline{X}$ or $\overline{T}$ respectively. Then $\operatorname{Irr}(\beta) = \{\mu^{(X)}\phi : \phi \in \operatorname{Irr}(\overline{\beta})\}$, identifying characters in $\overline{\beta}$ with their inflations to characters of X. A similar description of $\operatorname{Irr}(\beta_i)$, in terms of $\mu^{(T)}$ and $\operatorname{Irr}(\overline{\beta_i})$, may be given for each i. Restricting irreducible characters in β to T, and doing likewise with $\overline{\beta}$ and $\overline{T}$, we deduce that the blocks of $\overline{T}$ covered by $\overline{\beta}$ are precisely the Fong correspondents $\overline{\beta_1}, \dots, \overline{\beta_n}$ of $\beta_1, \dots, \beta_n$ respectively.

Returning to the matter at hand, given a B-subpair (Q, b_0), we may use b_0^* (the unique block of $RQC_G(Q)$ lying over b_0) to define the Harris–Knörr correspondent block, $b_0^{*\prime}$, of $RNQC_G(Q)$ which lies over μ and has a defect group containing Q. Then $b_0^{*\prime}$ covers a unique block of $RNC_G(Q)$, say b_0', which we will (somewhat loosely) refer to as the Harris–Knörr correspondent of b_0. Let $\overline{b}_0$ denote the Fong correspondent of b_0'. By the digression above, $\overline{b}_0$ is the unique block of $RC_{\overline{G}}(\overline{Q})$ covered by the Fong correspondent of $b_0^{*\prime}$. This procedure gives us a subpair $(\overline{Q}, \overline{b}_0)$.

Let (D, b_1) be a maximal B-subpair. We now have a subpair $(\overline{D}, \overline{b}_1)$, which we claim is a maximal $\overline{B}$-subpair. We observe that when we apply the above procedure with (D, b_1), the block $\overline{b}_1$ has defect group $Z(\overline{D})$.

We take our previously defined block b for M and its Harris–Knörr correspondent b' for H. Then b lies over b_1 and b' lies over the Harris–Knörr correspondent, b_1', of b_1. The Fong correspondent of b' is $\overline{b}$, which we already know to be the unique Brauer correspondent of $\overline{B}$ for $N_{\overline{G}}(\overline{D})$. From the digression above, we conclude that the blocks of $RC_{\overline{G}}(\overline{D})$ covered by $\overline{b}$ are the Fong correspondents of the blocks of $RNC_G(D)$ covered by b'. Hence $\overline{b}_1$ is one of these blocks and $(\overline{D}, \overline{b}_1)$ is indeed a maximal $\overline{B}$-subpair.

Now we consider a more general chain σ, with maximal p-subgroup V^{σ}. Set $L_{\sigma} = NG_{\sigma}$ and recall that $\overline{L_{\sigma}} = \overline{G}_{\overline{\sigma}}$. We consider a block of RG_{σ}, say b_{σ}, which is a summand of B_{σ}. Let D_{σ} be a defect group for b_{σ} and recall that D_{σ} contains the normal p-subgroup, V^{σ}, of G_{σ} so that $D_{\sigma}C_G(D_{\sigma}) \leq G_{\sigma}$. We choose a maximal b_{σ}-subpair, (D_{σ}, b_2), which may also be viewed as a B-subpair. We note that $Z(D_{\sigma})$ is the unique defect group for b_2. We assume (as we may, after possibly amending our original choice of D) that (D, b_1) contains (D_{σ}, b_2).

There is a B-subpair $(D_{\sigma}C_D(D_{\sigma}), b_{\infty})$ contained in (D, b_1) and containing (D_{σ}, b_2). This too may be regarded as a b_{σ}-subpair. Consequently, we have $C_D(D_{\sigma}) \leq D_{\sigma}$ by the maximality of (D_{σ}, b_2) as a b_{σ}-subpair.

From now on, we regard (D_{σ}, b_2) as a B-subpair. As above, we construct from this a subpair $(\overline{D_{\sigma}}, \overline{b}_2)$. Now $\overline{G}_{\overline{\sigma}}$ contains $\overline{D_{\sigma}}C_{\overline{G}}(\overline{D_{\sigma}})$, so that $\overline{b}_2$ has a unique Brauer correspondent block for $\overline{G}_{\overline{\sigma}}$, which we denote by $\overline{b}_{\overline{\sigma}}$.

Let b'_{σ} be the unique Harris–Knörr correspondent of b_{σ} for RL_{σ}. The earlier argument (for G, H, and B) may be applied with L_{σ} in the role of G and b'_{σ} in the role of B. We deduce that the unique Brauer correspondent of $\overline{b}_2$ for $\overline{G}_{\overline{\sigma}}$ is the Fong correspondent of b'_{σ}. In other words, $\overline{b}_{\overline{\sigma}}$ is the Fong correspondent of b'_{σ}. We wish to know that $\overline{b}_{\overline{\sigma}}$ is a Brauer correspondent of $\overline{B}$.

To prove this, it suffices to prove that $(\overline{D_{\sigma}}, \overline{b}_2)$ is a $\overline{B}$-subpair. Since $(\overline{D}, \overline{b}_1)$ is a $\overline{B}$-subpair, it suffices to prove that $(\overline{D_{\sigma}}, \overline{b}_2)$ is contained in $(\overline{D}, \overline{b}_1)$.

If possible, choose a B-subpair (D_3, b_3), contained in (D, b_1) and containing (D_{σ}, b_2), which is maximal subject to the subpair $(\overline{D_3}, \overline{b}_3)$ not being contained in $(\overline{D}, \overline{b}_1)$. Note that $D_3 < D$ and that, furthermore, $C_D(D_3) = Z(D_3)$, as $D_{\sigma} \leq D_3$.

Let E_3 be a defect group for b_3. Then $E_3 \leq C_G(D_{\sigma})$ and certainly $Br_{E_3}(1_{b_2}) \neq 0$. Hence E_3 is contained in $Z(D_{\sigma})$, the unique defect group for b_2. Consequently, $E_3 \leq C_D(D_3) = Z(D_3)$, so that $E_3 = Z(D_3)$, as $Z(D_3)$ is contained in each defect group of every block of $RC_G(D_3)$.

We set $D_4 = N_D(D_3) > D_3$ and we work for a while within the group $X = NC_G(D_3)D_4$. Corresponding to the block b_3 is a block b''_3 of RX, lying over μ, having defect group D_4, which is the unique block of RX covering the Harris–Knörr correspondent, b'_3, or b_3 for $NC_G(D_3)$. By the earlier digression, the Fong correspondent of b''_3 (which has defect group $\overline{D_4}$) covers $\overline{b}_3$.

There is a B-subpair (D_4, b_4) lying between (D_3, b_3) and (D, b_1) and a corresponding subpair $(\overline{D_4}, \overline{b}_4)$. The block b''_3 has defect group D_4 and $(D_4,\ b_4)$ may be viewed as a maximal b''_3-subpair containing the b''_3-subpair (D_3, b_3). Applying the earlier arguments (with X in the role of G and b''_3 in the role of B) and using the fact that the Fong correspondent of b''_3 lies over $\overline{b}_3$, we may conclude that $(\overline{D_4}, \overline{b}_4)$ contains $(\overline{D_3}, \overline{b}_3)$.

By the maximal choice of D_3, we know that $(\overline{D}, \overline{b}_1)$ contains $(\overline{D_4}, \overline{b}_4)$. This contradicts the fact that $(\overline{D}, \overline{b}_1)$ does not contain $(\overline{D_3}, \overline{b}_3)$. This contradiction shows that $(\overline{D_\sigma}, \overline{b}_2)$ is, after all, a $\overline{B}$-subpair.

The conclusion of this part of the discussion is that the Fong correspondent of b'_σ is a Brauer correspondent of $\overline{B}$. On the other hand, given a block $\overline{b}_{\overline{\sigma}}$ which is a summand of $\overline{B}_{\overline{\sigma}}$, by essentially applying the above arguments in reverse we see that $\overline{b}_{\overline{\sigma}}$ occurs as a Fong correspondent of one of the block summands of B'_σ.

Returning to our proof of the main theorem, we obtain (for our fixed "bad" defect d)

$$\sum_{\sigma \in \mathcal{N}(G,Z)/G} (-1)^{|\sigma|} k_d(B_\sigma) = \sum_{\tilde{\sigma} \in \mathcal{N}(\tilde{G}, \tilde{Z})/\tilde{G}} (-1)^{|\sigma|} k_d\left(\tilde{B}_{\tilde{\sigma}}\right).$$

If $O_p(\tilde{G}) > \tilde{Z}$, then the right-hand alternating sum vanishes by the usual elementary argument. Since we know that the left-hand alternating sum does not vanish, we conclude that $O_p(\tilde{G}) = \tilde{Z}$. But then, by the minimal choice of G, the right-hand alternating sum vanishes, since it is that which appears in the "weak" form of Dade's projective conjecture for $\tilde{B}$. This contradiction finally shows that μ is not G-stable.

For the remainder of the proof, it is convenient to deal with the conjecture via B-subpairs. Instead of working with $\mathcal{N}(G,Z)$, we work with $\mathcal{N}_B(G,Z)$, whose chains have the form

$$(\sigma, b) = \left((Q_0, b_0) < \cdots < (Q_n, b_n)\right),$$

where the p-subgroups appearing form a chain, σ, in $\mathcal{N}(G,Z)$ and, for each Q_i, the subpair (Q_i, b_i) is the unique B-subpair contained in the B-subpair (V^σ, b) $(= (Q_n, b_n))$. There is a natural conjugation action of G on $\mathcal{N}_B(G,Z)$. Note that the notation we adopt is justified by the fact that once the chain $\sigma \in \mathcal{N}(G,Z)$ and the block, b, of $RC_G(V^\sigma)$ are specified, the chain of subpairs is uniquely determined. Furthermore, this uniqueness shows that the stabilizer $G_{(\sigma,b)}$ is precisely $I_{G_\sigma}(b)$, the inertial subgroup of b in G_σ. We observe that 1_b is also a centrally primitive idempotent of $RG_{(\sigma,b)}$, and we let $\beta_{(\sigma,b)}$ denote the block $1_b.RG_{(\sigma,b)}$.

For any $\sigma \in \mathcal{N}(G,Z)$, the blocks which occur as summands of B_σ are in bijection with the orbits of G_σ on B-subpairs of the form (V^σ, b), as $V^\sigma C_G(V^\sigma) \lhd G_\sigma$. Furthermore, given a choice of b, the block of RG_σ induced (characterwise) from $\beta_{(\sigma,b)}$ is the block summand of B_σ lying over b. We note that, for any positive integer d, this induced block contributes $(-1)^{|\sigma|} k_d(\beta_{(\sigma,b)})$ towards $(-1)^{|\sigma|} k_d(B_\sigma)$, as its irreducible characters are in defect-preserving bijection with the irreducible characters of $\beta_{(\sigma,b)}$.

For any positive defect d' we may calculate

$$\sum_{\sigma \in \mathcal{N}(G,Z)/G} (-1)^{|\sigma|} k_{d'}(B_\sigma)$$

by counting (for all $\sigma \in \mathcal{N}(G,Z)/G$) the contribution from G_σ-orbits of B-subpairs of the form (V^σ, b). By the discussion above, this is the same as taking the appropriate alternating sum over all $(\sigma, b) \in \mathcal{N}_B(G,Z)/G$.

Hence the "weak" form of Dade's projective conjecture may be equivalently stated as the requirement that

$$\sum_{(\sigma,b) \in \mathcal{N}_B(G,Z)/G} (-1)^{|\sigma|} k_{d'}(\beta_{(\sigma,b)}) = 0$$

for each positive defect d'.

The inertial subgroup $I_G(\mu) = I$ satisfies $[I : Z(I)] < [G : Z(G)]$, as I contains $Z(G)$ and μ is not G-stable. Let B' be the block of RI which lies over μ and whose irreducible characters induce those of B. There is a bijection between $\mathcal{N}_B(G,Z)/G$ and $\mathcal{N}_{B'}(I,Z)/I$ which we now describe. We choose (as we may) a maximal B-subpair (D, b_D) such that the Glauberman correspondent $f_D(\mu)$ lies under b_D. (Implicit here is the assumption that $D \leq I$. Then D is a defect group for both B and B'.) For each subgroup, Q, of D, we let (Q, b_Q) denote the unique B-subpair contained in (D, b_D). Such subpairs include a full set of representatives for the G-conjugacy classes of B-subpairs. For each subgroup, Q, of D, the Glauberman correspondent $f_Q(\mu)$ lies under b_Q.

We wish to replace (Q, b_Q) by a B'-subpair (Q, b'_Q). We obtain b'_Q by multiplying the block idempotent of b_Q by $e_{f_Q(\mu)}$, which is the centrally primitive idempotent of $RC_N(Q)$ associated to $f_Q(\mu)$. This yields a centrally primitive idempotent of $RC_I(Q)$ (it is easy to check that $C_I(Q)$ is the inertial subgroup in $C_G(Q)$ of $f_Q(\mu)$), and b'_Q is defined as the corresponding block of $RC_I(Q)$. To check that this really is a B'-subpair, we need to know that $Br_Q(1_{B'})$ involves $Br_Q(1_{b'_Q})$ (this is just the image of $1_{b'_Q}$ (mod $J(R)G$)).

Since $1_{B'} = 1_B e_\mu$ and $Br_D(1_{b_D})$ appears in $Br_D(1_B)$, we have what we wanted when $Q = D$ (applying Br_D to both sides, then multiplying by $Br_D(1_{b_D})$) since we assumed that $e_{f_D(\mu)}.1_{b_D} \neq 0$. For $Q \lhd D$, we have $Br_D(e_{f_D(\mu)}.1_{b_D}) = Br_D(1_{b_D}).Br_D(e_{f_Q(\mu)}.1_{b_Q})$, so certainly $e_{f_Q(\mu)}.1_{b_Q} \neq 0$. Applying Br_Q to the equation $1_{B'} = 1_B e_\mu$, then multiplying by $Br_Q(1_{b_Q})$, we obtain what we wanted when $Q \lhd D$. Repeating this argument enough times gives the desired involvement for all subgroups Q of D, as all such subgroups are subnormal in D.

For any subgroup, Q, of D, the argument above also shows that $(Q, b'_Q) \leq (D, b'_D)$. Hence we have a canonical bijection between B-subpairs contained in (D, b_D) and B'-subpairs contained in (D, b'_D), which extends to a canonical bijection between chains (σ, b) with $(V^\sigma, b) \leq (D, b_D)$ and chains (σ', b') with $(V^{\sigma'}, b') \leq (D, b'_D)$.

We note that we have $N_G(Q, b_Q) \cap I = N_I(Q, b'_Q)$ and that $\mathrm{Tr}_{C_I(Q)}^{C_G(Q)}(1_{b'_Q}) = 1_{b_Q}$. From this we see that $N_G(Q, b_Q) = C_G(Q)N_I(Q, b'_Q)$.

Suppose that (Q, b_Q) and $(Q, b_Q)^g$ are both contained in (D, b_D) for some $g \in G$. We claim that the corresponding B'-subpairs (Q, b'_Q) and (Q^g, b'_{Q^g}) are I-conjugate. For $b_{Q^g} = (b_Q)^g$ by the uniqueness of $(Q^g, b_{Q^g}) \leq (D, b_D)$. The Glauberman correspondent $f_{Q^g}(\mu^g)$ (of the Q^g-stable character, μ^g, of N) clearly lies under $(b_Q)^g$. Since all irreducible characters of $C_N(Q^g)$ which lie under b_{Q^g} are conjugate via elements of $C_G(Q^g)$ to $f_{Q^g}(\mu)$, we deduce that μ may be conjugated to μ^g by an element of $C_G(Q^g)$. Hence there are elements $x \in I$ and $c \in C_G(Q)$ with $g = cx$. Now we have $1_{b'_{Q^x}} = e_{f_{Q^x}(\mu)}(1_{b_Q})^x = 1_{(b'_Q)^x}$, as $\mu^x = \mu$. Since $Q^x = Q^g$, the claim is established.

On the other hand, if (Q, b'_Q) and $(Q, b'_Q)^x$ are B'-subpairs which are both contained in (D, b'_D) for some $x \in I$, then (Q, b_Q) and $(Q, b_Q)^x$ are B-subpairs which are both contained in (D, b_D). First, uniqueness subject to containment in (D, b'_D) tells us that $b'_{Q^x} = (b'_Q)^x$. Since

$$\left(1_{b'_Q}\right)^x 1_{b'_{Q^x}} = \left(e_{f_Q(\mu)} 1_{b_Q}\right)^x 1_{b_{Q^x}} e_{f_{Q^x}(\mu)} \neq 0,$$

we certainly have $(1_{b_Q})^x 1_{b_{Q^x}} \neq 0$, so that $b_{Q^x} = (b_Q)^x$.

More generally, given $g \in G$ and a chain (σ, b_{V^σ}) such that (V^σ, b_{V^σ}) and $(V^\sigma, b_{V^\sigma})^g$ are both contained in (D, b_D), we may write $g = cx$ for some $x \in I$ and some $c \in C_G(V^\sigma)$. This implies, in particular, that $(\sigma, b_{V^\sigma})^g = (\sigma, b_{V^\sigma})^x$ and then that the corresponding chains (σ, b'_{V^σ}) and $(\sigma^g, b'_{(V^\sigma)^g})$ are I-conjugate (via x). Conversely, given a chain (σ, b'_{V^σ}) and an element $x \in I$, with both (σ, b'_{V^σ}) and $(\sigma, b'_{V^\sigma})^x$ contained in (D, b'_D), we conclude from the earlier argument that the corresponding chains (σ, b_{V^σ}) and $(\sigma, b_{V^\sigma})^x$ are both contained in (D, b_D).

We have now demonstrated that the canonical correspondence between chains (σ, b) with $(V^\sigma, b) \leq (D, b_D)$ and chains (σ', b') with $(V^{\sigma'}, b') \leq (D, b'_D)$ induces a bijection between $\mathscr{N}_B(G, Z)/G$ and $\mathscr{N}_{B'}(I, Z)/I$.

Furthermore, for corresponding chains (σ, b) and (σ, b'), we have $G_{(\sigma, b)} \cap I = I_{(\sigma, b')}$. There is a defect-preserving bijection between the irreducible characters in $\beta_{(\sigma, b)}$ and in the analogously defined block $\beta'_{(\sigma, b')}$, since $I \cap G_{(\sigma, b)}$ is the inertial subgroup of $f_{V^\sigma}(\mu)$ in $G_{(\sigma, b)}$.

We thus obtain (for our fixed "bad" defect d)

$$\sum_{(\sigma, b) \in \mathcal{N}_B(G, Z)/G} (-1)^{|\sigma|} k_d(\beta_{(\sigma, b)}) = \sum_{(\sigma', b') \in \mathcal{N}_{B'}(I, Z)/I} (-1)^{|\sigma'|} k_d(\beta'_{(\sigma', b')}).$$

If $O_p(I) > Z$, then the usual elementary argument shows that the right-hand alternating sum is zero, a contradiction, as the left-hand alternating sum does not vanish by hypothesis. Hence $O_p(I) = Z$.

By the minimal choice of G, the formula of Dade's projective conjecture holds for B', for each linear character of Z and each positive defect. Hence the "weak" form of Dade's projective conjecture and its subpair version both hold for B', for each positive defect. Thus the right-hand alternating sum above is zero, a contradiction. The proof of Theorem 1 is complete.

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